King Fahd University of Petroleum & Minerals
College of Computer Sciences & Engineering
Department of Information and Computer Science

ICS 253: Discrete Structures I

Section 03 – Dr. Husni Al-Muhtaseb

Final Exam – 142

120 Minutes

Calculators, mobile phones and other electronic devices are not permitted.

Question	Max	Earned	CLO*	Question	Max	Earned	CLO*
1	5		1	12	4		3
2	5		1	13	4		3
3	5		1	14	4		3
4	5		2	15	4		3
5	5		2	16	4		3
6	5		2	17	4		3
7	8		2	18	4		3
8	4		3	19	4		3
9	4		3	20	6		3
10	4		3	21	5		1
11	2		3	22	5		1

* **CLO** Course Learning Outcomes

Total Out 100	

Thursday, May 21, 2015

Sample Solution

Question 1: [5 Points] Logic and Proofs

Select the Boolean expression that is logical equivalent to the expression $((p \vee q) \wedge \neg p \wedge r) \vee r)$ form the following. Circle the correct answer.

(a) $p \wedge r$ (b) $p \vee q$ (c) r (d) p

Question 2: [5 Points] Logic and Proofs

Circle the correct answer. Select one statement of the following statements that is equivalent to the statement

"If n is divisible by 36, then n is divisible by 4 and by 9."

(a) If n is divisible by 4 and divisible by 9 then n is divisible by 36.

(b) If n is not divisible by 4 or not divisible by 9 then n is not divisible by 36.

(c) If n is not divisible by 36 then n is divisible by 4 or divisible by 9.

(d) If n is not divisible by 36 then n is not divisible by 4 or not divisible by 9.

Question 3: [5 Points] Logic and Proofs

What is the contrapositive statement of the statement,
"If the angle between a and b is 90 degrees, then $a^2 + b^2 = c^2$"

If $a^2 + b^2 \neq c^2$ then the angle between a and b is not 90 degrees.

Question 4: [5 Points] Functions

Define $f: \mathbb{R} \to \mathbb{Z}$, $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{R}$ and $f(x) \in \mathbb{Z}$. where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}$ is the set of all integer numbers. Determine if f is a function. If it is not a function show why. If it is a function determine if it is onto and/or one-to-one. Note: $\lfloor x \rfloor$ is the floor of x.

Function. Not one-to-one. Onto.

Question 5: [5 Points] Sequences and Summations

Describe the sequences $0, \frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \frac{24}{25}, \frac{35}{36}, \ldots$ by an explicit formula.

$$\frac{n^2 - 1}{n^2}$$

Question 6: [5 Points] Induction

We are going to prove by induction that for all integers $k \geq 1$, $\sqrt{k} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}}$. Clearly this is true for $k = 1$. Assume the Induction Hypothesis (IH) that

$$\sqrt{n} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

Which is a correct way of concluding this proof by induction?

(a) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \sqrt{n+1} + 1 \geq \sqrt{n+1}$

(b) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}$

(c) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}} = \frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}} \geq \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n+1}} = \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$

(d) By IH, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n}} = \frac{\sqrt{n}\sqrt{n}+1}{\sqrt{n}} \geq \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1}$

Question 7: [8 Points] Induction and Recursion

We are going to use structural induction to show that if T is a full binary tree, then $l(T) = i(T) + 1$. Where $l(T)$ is the number of leaves of T and $i(T)$ is the number of internal vertices of T. Note: Leaves are nodes with no children.

Base Step: (T = leaf). We have $l(\text{leaf}) = 1 = 1 + 0 = 1 + i(\text{leaf})$

Induction step: Assume $l(T_1) = 1 + i(T_1)$ and $l(T_2) = 1 + i(T_2)$, where T_1 and T_2 are full binary trees and ($T = T_1 \cdot T_2$).

Complete the remaining parts of the proof.

We have

$l(T_1 \cdot T_2) = l(T_1) + l(T_2)$ by the def. of l

$= 1 + i(T_1) + l(T_2)$ by the IH for T_1

$= 1 + i(T_1) + 1 + i(T_2)$ by the IH for T_2

$= 1 + i(T_1 \cdot T_2)$ by the def. of i

We have completed all the cases for structural induction on a binary tree, so we can therefore conclude that for any full binary tree T, $l(T) = 1 + i(T)$.

Question 8: [4 Points] Counting and the Pigeonhole Principle

Show that among any 4 different numbers one can find 2 numbers so that their difference is divisible by 3. (Avoid considering the cases separately. Use Pigeonhole Principle!)

There are 3 possible remainders when we divide a number by 3 (0, 1, or 2). Thus by PHP, since we have 4 numbers, some two of them must have the same remainder on division by 3—so we can write these two as

$n_1 = 3k_1 + r$ and $n_2 = 3k_2 + r$,

where r is the remainder on division by 3. Then the difference is

$n_1 - n_2 = (3k_1 + r) - (3k_2 + r)$

$= 3k_1 + r - 3k_2 - r$

$= 3k_1 - 3k_2$

$= 3(k_1 - k_2)$,

which is divisible by 3.

Question 9: [4 Points] Permutations and Combinations

How many positive decimal integers less than 1000 are divisible by either 7 or 11?

Since 1000 = (11)(90) + 10, there are 90 multiples of 11 less than 1000. Now, if we add the 142 multiples of 7 to this, we get 232, but in doing this we've counted each multiple of 77 twice. We can correct for this by subtracting off the 12 items that we've counted twice. Thus, there are 232-12=220 positive integers less than 1000 divisible by 7 or 11.

Question 10: [4 Points] Permutations and Combinations

A group of people is going to sit at a round table. Two arrangements will be considered the same if each person has the same person to the right and the same person to the left. For example, if everybody got up and moved one seat to the right, that would be considered the same arrangement. All each person cares about is who is to the left and who is to the right. There are twelve chairs at a round table. Twelve people are going to be seated. How many arrangements are possible?

The first person can sit anywhere. After that, there are 11 people who can sit to the right of that first person. There are 10 people who can sit to the right of the second person, etc. The total number of ways is 11! = 39916800

Question 11: [2 Points] Probability

The random variable X on a sample space S = {1, 2, 4, 10} has the following distribution:

X	1	2	4	10
P(x)	0.4	0.3	0.2	?

What is P(X = 10)?

1 – (0.4+0.3+0.2) = 0.1

Question 12: [4 Points] Probability

Two fair dice are rolled, find the probability that the sum is less than 13.

The sample space S of two dice is shown below.

S = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6), (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) }

The event E is the same as the sample space E=S

P(E) = n(E)/n(S) = 36/36 = 1

Question 13: [4 Points] Probability

A machine produces parts that are either good (90%), slightly defective (2%), or obviously defective (8%). Produced parts get passed through an automatic inspection machine, which is able to detect any part that is obviously defective and discard it. What is the quality of the parts that make it through the inspection machine and get shipped?

Hint: Compute the probability that a part is good given that it is not obviously defective.

Let

G be the event that a randomly chosen shipped part is good,

S be the event that a randomly chosen shipped part is slightly defective,

O the event that a randomly chosen shipped part is obviously defective

We are told that P(G) = .90, P(S) = 0.02, and P(O) = 0.08.

We want to compute the probability that a part is good given that it passed the inspection machine (i.e., it is not obviously defective), which is

P(G|O) = P(G ∩ O) / P(O) = P(G) / (1 − P(O)) = .90 / (1 − .08) = 90/92 = .978

$$\boldsymbol{P(G|\bar{O}) = \frac{P(G\cap\bar{O})}{P(\bar{O})} = \frac{P(G)}{1-P(O)} = \frac{0.9}{1-0.08}=0.978}$$

Question 14: [4 Points] Sets

Which of the following statements is FALSE?

(a) $2 \in A \cup B$ implies that if $2 \notin A$ then $2 \in B$.

(b) $\{2, 3\} \subseteq A$ implies that $2 \in A$ and $3 \in A$.

(c) $\{2\} \in A$ and $\{3\} \in A$ implies that $\{2, 3\} \subseteq A$.

(d) $A \cap B \supseteq \{2, 3\}$ implies that $\{2, 3\} \subseteq A$ and $\{2, 3\} \subseteq B$.

Question 15: [4 Points] Recursive Definition

A bit string that reads the same backwards as forwards is called a palindrome. That is a palindrome is a string which, when reversed it is identical to the original string, eg. 0110 is a palindrome and 0100 is not. More examples of bit string palindromes are: 0, 1, 00, 11, 010, 101, 000, 111, 1111, 10101, 01010, ...

Give a recursive definition of the set of bit strings that are palindrome.

Basis Step:

$\lambda \in S$ (λ is the empty string)

$0 \in S$

$1 \in S$

Recursive Step: If $x \in S$, then $0x0 \in S$ and $1x1 \in S$

Question 16: [4 Points] Linear Recurrence Relations

A piece of paper is 2 unit thick. By folding into half, the thickness becomes 4 units. Folding into half again, its thickness becomes 8 units, and so on. What is the thickness of the paper after it is folded 20 times?

When it is folded 3 times, the thickness becomes 16 units. Similarly, when folded 4, 5, 6 times, the thickness becomes 32, 64 and 128 units respectively. You notice that each time the paper is folded, its thickness doubles, so you just multiply the thickness by 2 each time. Continuing in the same way, it is easy to get the answer, namely, when folded 20 times, the thickness will become $2 \times 2^{20} = 2^{21}$ units.

Question 17: [4 Points] Binomial Coefficients and Identities

What is the coefficient of $x^{88}y^{62}$ in the expansion of $(4x - 5y)^{150}$? Show your answer using combination and multiplication of numbers (no need to calculate a final answer).

$$\binom{150}{62}(4x)^{88}(-5y)^{62} = \binom{150}{62}4^{88}(-5)^{62}x^{88}y^{62}$$

$$\binom{150}{62}4^{88}(-5)^{62}$$

Question 18: [4 Points] Binomial Coefficients and Identities

The row of Pascal's triangle containing the binomial coefficients $\binom{8}{k}$, $0 \le k \le 8$, is:

1 8 28 56 70 56 28 8 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

1 9 36 84 126 126 84 36 9 1

Question 19: [4 Points] Linear Recurrence Relations

Which of the following is a linear homogeneous recurrence relation with constant coefficients? Circle all answers.

(a) $a_n = a_{n-1} - 4a_{n-2} + a_{n-3}^3$

(b) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$

(c) $a_n = 3a_{n-2}$

(d) $a_n = \frac{a_{n-5}}{n}$

Question 20: [6 Points] Linear Recurrence Relations

Solve the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \ge 2$ given the initial conditions $a_0 = 1$ and $a_1 = 1$.

$c_1 = -2$, $c_2 = -3$,

$r^2 - 2r - 3 = 0$

=> $r_1 = 3$ $r_2 = -1$

=> $a_n = \alpha_1 * 3^n + \alpha_2 * (-1)^n$

with the initial condition: $a_0 = 1$, $a_1 = 1$

=> $1 = \alpha_1 * 3^0 + \alpha_2 * (-1)^0 = \alpha_1 + \alpha_2$

$1 = \alpha_1 * 3^1 + \alpha_2 (-1)^1 = 3\alpha_1 - \alpha_2$

=> $\alpha_1 = 1/2$, $\alpha_2 = 1/2$

=> $a_n = 1/2 * 3^n + 1/2 * (-1)^n$

Question 21: [5 Points] Logic and Proofs

The statement form (p ↔ r) → (q ↔ r) is equivalent to

(a) ~[(~p ∨ r) ∧ (p ∨ ~r)] ∨ [(~q ∨ r) ∧ (q ∨ ~r)]

(b) [(~p ∨ r) ∧ (p ∨ ~r)] ∨ ~[(~q ∨ r) ∧ (q ∨ ~r)]

(c) ~[(~p ∨ r) ∧ (p ∨ ~r)] ∧ [(~q ∨ r) ∧ (q ∨ ~r)]

(d) [(~p ∨ r) ∧ (p ∨ ~r)] ∧ [(~q ∨ r) ∧ (q ∨ ~r)]

Question 22: [5 Points] Logic and Proofs

Let m = "Ali is a math major," c = "Ali is a computer science major,"
g = "Ali's friend is a physics major," h = "Ali's friend has visited Dubai," and
t = "Ali's friend has visited Oman."

Which of the following expresses the statement "Ali is a computer science major and a math major, but his friend is a physics major who has not visited both Dubai and Oman."

(a) c ∧ m ∧ (g ∨ (~h ∨ ~t))

(b) c ∧ m ∧ g ∧ (~h ∧ ~t)

(c) c ∧ m ∧ (g ∨ (~h ∧ ~t))

(d) c ∧ m ∧ g ∧ (~h ∨ ~t)